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Analytic solutions of carrier flow equations via the Painlevé analysis approach

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Abstract. In this paper, we use the Painlevé approach to analyse a system of partial differential equations governing the carrier flow in semiconductor devices and obtain its auto-Bäcklund transformations. Moreover, we obtain some analytic solutions directly from the Painlevé-Bäcklund equations by introducing two elementary homographic invariants.

1. Introduction

The carrier flow in semiconductor devices is described by a system of nonlinear partial differential equations. Let u, v and w be the density of conduction band electrons, the density of valence band holes and the electrostatic potential, respectively. Let R denote the net recombination depending on u and v. Then we have

$$\nabla \cdot (c \nabla w) = u - v - N \tag{1.1}$$

$$u_t = \nabla \cdot (\mu_u (\nabla u - u \nabla w)) - R(u, v) \tag{1.2}$$

$$v_t = \nabla \cdot (\mu_v (\nabla v + v \nabla w)) - R(u, v) \tag{1.3}$$

where c is the dielectric constant, $N = N_D - N_A$, N_D and N_A being the densities of donor and acceptor ions respectively. μ_u and μ_v are the electron and hole mobilities respectively. Generally, μ_u and μ_v depend on N_D, N_A and ∇w.

In practical problems, u, v and w are defined on a connected, bounded open domain. Thus we have to consider (1.1)-(1.3) subject to suitable initial and boundary conditions. Some authors have studied several simplified cases and proved the existence and regularity of the unique solution, while others have solved it numerically (see e.g. Mock [1] and Kuo Pen-yu [2]). On the other hand, for theoretical reasons, we are also interested in analytic solutions.

As we know, Weiss, Tabor and Carnevale developed a creative method, called the WTC method, with successful applications to single partial differential equations [3, 4]. Grauel [5] used this method for some systems of nonlinear ordinary differential equations. Clearly, it is not easy to generalise this method to systems of nonlinear partial differential equations.

The aim of this paper is to look for analytic solutions of (1.1)-(1.3) via the Painlevé analysis. In the following section, we perform the Painlevé analysis to show that this system has no Painlevé property for partial differential equations, and thus it is not an integrable system in the sense of Weiss *et al* [3, 4]. We still use this method, however,

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to get some useful information by introducing the 'log' term in the expansion. Therefore, we get an auto-Bäcklund transformation naturally. In section 3, some non-trivial analytic solutions are obtained directly following the ideas of Conte and Musette [6, 7]. The final section is devoted to further discussion.

2. Painlevé analysis

In some practical cases, we can simplify the model (1.1)-(1.3). It means that μ_u and μ_v are constants and $R(u, v) = 0$. For simplicity of analysis, we only consider this case and put $\mu_u = a$ and $\mu_v = b$. Then (1.1)-(1.3) become

$$c\Delta w = u - v - N \tag{2.1}$$

$$u_t = a(\Delta u - vu \cdot \nabla w) - au(u - v - N)/c \tag{2.2}$$

$$v_t = b(\Delta v + \nabla v \cdot \nabla w) + bv(u - v - N)/c. \tag{2.3}$$

The main idea of the WTC method is to demonstrate that the solutions comprising the 'ansatze'

$$w = \sum_{j=0}^{\infty} w_j M^{j-p_1} \tag{2.4}$$

$$u = \sum_{k=0}^{\infty} u_k M^{k-p_2} \tag{2.5}$$

$$v = \sum_{l=0}^{\infty} v_l M^{l-p_3} \tag{2.6}$$

are single valued about the singularity manifold $M = 0$; that is, p_1, p_2 and p_3 are positive integers, M is analytic and non-characteristic ($M_t M_x \neq 0$) and all recursion relations for w_j, u_k and v_l are self-consistent. By substituting (2.4)-(2.6) into (2.1)-(2.3) and analysing the order of leading parts, we obtain

$$p_1 = 0 \qquad p_2 = p_3 = 2.$$

Since $p_1 = 0$ is not allowed, the system (2.1)-(2.3) is not integrable in the sense of the Painlevé property for partial differential equations. Accordingly, we introduce a 'log' term in (2.4) and adopt the following expansions:

$$w = T \log M + w_0 \tag{2.7}$$

$$u = u_0/M^2 + u_1/M + u_2 \tag{2.8}$$

$$v = v_0/M^2 + v_1/M + v_2. \tag{2.9}$$

We require that w_0, u_2 and v_2 satisfy the original system and so (2.7)-(2.9) are just auto-Bäcklund transformations provided the above form it not contradictory.

We substitute (2.7)-(2.9) into (2.1)-(2.3) and obtain the following equations:

$$u_0 - v_0 = -cT|\nabla M|^2 \tag{2.10}$$

$$u_1 - v_1 = c(2\nabla T \cdot \nabla M + T\Delta M) \tag{2.11}$$

$$u_2 - v_2 = c\Delta T \log M + c\Delta W_0 + N \tag{2.12}$$

$$[(2T + 6)|\nabla M|^2 - (u_0 - v_0)/c]u_0 = 0 \tag{2.13}$$

$$-2u_0M_t/a=(-4\nabla u_0\cdot\nabla M-2u_0\Delta M+2u_1|\nabla M|^2)$$
$$+(2u_0\nabla w_0\cdot\nabla M+Tu_1|\nabla M|^2-T\nabla M\cdot\nabla u_0)$$
$$-2u_0u_1/c+(u_1v_0+u_0v_1)/c+(2u_0\nabla T\cdot\nabla M)\log M \tag{2.14}$$

$$(u_{0,t}-u_1M_t)/a$$
$$=(\Delta u_0-2\nabla u_1\cdot\nabla M-u_1\Delta M)-(\nabla u_0\cdot\nabla w_0-u_1\nabla M\cdot\nabla w_0+T\nabla M\cdot\nabla u_1)$$
$$-(u_1^2+2u_0u_2)/c+(u_1v_1+u_2v_0+u_0v_2)/c+Nu_0/c$$
$$-[(\nabla u_0-u_1\nabla M)\cdot\nabla T]\log M \tag{2.15}$$

$$u_{1,t}/a=\Delta u_1-(T\nabla M\cdot\nabla u_2+\nabla u_1\cdot\nabla W_0)-2u_1u_2/c$$
$$+(u_1v_2+v_1u_2)/c+Nu_1/c-(\nabla T\cdot\nabla u_1)\log M \tag{2.16}$$

$$u_{2,t}=a(\Delta u_2-\nabla u_2\cdot\nabla w_0)-au_2(u_2-v_2-N)/c-(a\nabla T\cdot\nabla u_2)\log M \tag{2.17}$$

$$[(-2T+6)|\nabla M|^2+(u_0-v_0)/c]v_0=0 \tag{2.18}$$

$$-2v_0M_t/b=(-4\nabla v_0\cdot\nabla M-2v_0\Delta M+2v_1|\nabla M|^2)$$
$$-(2v_0\nabla w_0\cdot\nabla M+Tv_1|\nabla M|^2-T\nabla M\cdot\nabla v_0)-2v_0v_1/c$$
$$+(u_0v_1+u_1v_0)/c-2(v_0\nabla T\cdot\nabla M)\log M \tag{2.19}$$

$$(v_{0,t}-v_1M_t)/b$$
$$=(\Delta v_0-2\nabla v_1\cdot\nabla M-v_1\Delta M)+(\nabla v_0\cdot\nabla w_0-v_1\nabla M\cdot\nabla w_0+T\nabla M\cdot\nabla v_1)$$
$$-(v_1^2+2v_0v_2)/c+(u_1v_1+u_0v_2+u_2v_0)/c$$
$$-Nv_0/c+[(\nabla v_0-v_0\nabla M)\cdot\nabla T]\log M \tag{2.20}$$

$$v_{1,t}/b=\Delta v_1+(T\nabla M\cdot\nabla v_2+\nabla v_1\cdot\nabla w_0)-2v_1v_2/c+(u_1v_2+v_1u_2)/c$$
$$-Nv_1/c+(\nabla T\cdot\nabla v_1)\log M \tag{2.21}$$

$$v_{2,t}=b(\Delta v_2+\nabla v_2\cdot\nabla w_0)+bv_2(u_2-v_2-N)/c+(b\nabla T\cdot\nabla v_2)\log M. \tag{2.22}$$

Because w_0, u_2 and v_2 satisfy (2.1)–(2.3), we have from (2.12), (2.17) and (2.22) that

$$T=\nabla T\cdot\nabla u_2=\nabla T\cdot\nabla v_2=0. \tag{2.23}$$

Clearly, the above statements tell us that there is a special solution $T=T(t)$, where $T(t)$ is an arbitrary function of t. We might as well express the solution in the more general form $T(x, t, u_2, v_2)$. Furthermore, (2.13) and (2.18) lead to the following four systems:

system I: $u_0\neq 0$, $v_0\neq 0$ and

$$(2T+6)|\nabla M|^2-(u_0-v_0)/c=0$$
$$(-2T+6)|\nabla M|^2+(u_0-v_0)/c=0$$

system II: $u_0=0$, $v_0\neq 0$ and

$$(-2T+6)|\nabla M|^2-v_0/c=0$$

system III: $u_0 \neq 0$, $v_0 = 0$ and

$$(2T+6)|\nabla M|^2 - u_0/c = 0$$

system IV: $u_0 = 0$ and $v_0 = 0$.

We first analyse system I and find out that

$$|\nabla M|^2 = 0 \tag{2.24}$$

$$u_0 = v_0. \tag{2.25}$$

Thus (2.11) becomes

$$u_1 = v_1 \tag{2.26}$$

and (2.10) is satisfied automatically. Therefore (2.14) tells us that $M_t = 0$ and so M is constant, say $M = M_0$. It is easy to verify that (2.19) holds automatically. Moreover, we have from (2.15) that

$$u_{0,t}/a = \Delta u_0 - \nabla u_0 \cdot \nabla w_0 - u_0 \Delta w_0 - (\nabla u_0 \cdot \nabla T) \log M_0.$$

Similarly, (2.20) becomes

$$v_{0,t}/b = \Delta v_0 + \nabla v_0 \cdot \nabla w_0 + v_0 \Delta w_0 + (\nabla v_0 \cdot \nabla T) \log M_0.$$

Since $u_0 = v_0$, the above two equations lead to

$$v_{0,t} = 2ab \Delta v_0/(a+b) \tag{2.27}$$

$$\Delta v_0 = (a+b)(\nabla \cdot (v_0 \nabla w_0) + (\nabla v_0 \cdot \nabla T) \log M_0)/(a-b). \tag{2.28}$$

Similarly to (2.15) and (2.20), equations (2.16) and (2.21) also give us the following equations:

$$v_{1,t} = 2ab \Delta v_1/(a+b) \tag{2.29}$$

$$\Delta v_1 = (a+b)(\nabla \cdot (v_1 \nabla w_0) + (\nabla v_1 \cdot \nabla T) \log M_0)/(a-b). \tag{2.30}$$

Obviously, u_1 and v_1 satisfy the same equations as do u_0 and v_0. Since $M = M_0$ and (2.7)-(2.9), we obtain the auto-Bäcklund transformation

$$w = H + w_0 \tag{2.31}$$

$$u = F + u_2 \tag{2.32}$$

$$v = F + v_2 \tag{2.33}$$

where

$$H = T \log M_0,$$

$$F_t = 2ab \Delta F/(a+b) \tag{2.34}$$

and

$$\Delta F = (a+b)(\nabla \cdot (F \nabla w_0) + \nabla F \cdot \nabla H). \tag{2.35}$$

The simplest solution of (2.1)-(2.3) is $(c(t), N/2, -N/2)$, where $c(t)$ is an arbitrary function of t. On the other hand, if we take a simple solution of (2.23) to be an arbitrary function of t, say $T(t)$, then (2.31)-(2.33) lead to another solution $(c(t) + T(t) \log M_0, F_0 + N/2, F_0 - N/2)$, F_0 satisfying the Laplace equation $\Delta F = 0$ and $F_t = 0$ by means of (2.34)-(2.35).

Furthermore, by use of (2.23) and (2.31), we can integrate (2.35) into

$$\nabla F-(a+b)F\nabla w/(a-b)=\boldsymbol{f}(t) \tag{2.36}$$

where $\boldsymbol{f}(t)$ is an arbitrary vector function of t.

Now turning to system II, we get

$$v_0=c(-2T+6)|\nabla M|^2.$$

On the other hand, we have from (2.10) that

$$v_0=cT|\nabla M|^2$$

and so ($T=2$),

$$v_0=2c|\nabla M|^2. \tag{2.37}$$

Thus (2.14) gives us $u_1=0$ and (2.11) becomes

$$v_1=-2c\Delta M. \tag{2.38}$$

Therefore, (2.15) leads to $u_2=0$ and (2.16) holds automatically. Therefore, we obtain

$$w=2\log M+w_0 \tag{2.39}$$

$$u=0 \tag{2.40}$$

$$v=-2c\Delta(\log M)+v_2 \tag{2.41}$$

where $(w_0, 0, v_2)$ satisfy the original system of partial differential equations and M satisfies the following three equations:

$$M_t/b-\nabla M\cdot\nabla w_0+(\Delta M-\nabla|\nabla M|^2\cdot\nabla M/|\nabla M|^2)=0 \tag{2.42}$$

$$[2\nabla M\cdot\nabla M_t+(\Delta M)M_t]/b-[\nabla|\nabla M|^2+(\Delta M)\nabla M]\cdot\nabla w_0$$
$$+(2v_2/c+N/c-\Delta)|\nabla M|^2+(\Delta M)^2=0 \tag{2.43}$$

$$\Delta M_t/b-\nabla(\Delta M)\cdot\nabla w_0+(\nabla M\cdot\nabla+2\Delta M)v_2+(N/c-\Delta)\Delta M=0. \tag{2.44}$$

In the case of dimension 1×1, we verify the compatability of these equations. So (2.39)-(2.41) is another auto-Bäcklund transformation for the special case $u=0$.

System III is completely similar to system II. We obtain $T=-2$ and the auto-Bäcklund transformation

$$w=-2\log M+w_0 \tag{2.45}$$

$$u=-2c\Delta(\log M)+u_2 \tag{2.46}$$

$$v=0 \tag{2.47}$$

where $(w_0, u_2, 0)$ satisfy the original partial differential equations (2.1)-(2.3) and M satisfies

$$M_t/a+\nabla M\cdot\nabla w_0-(\Delta M-\nabla(|\nabla M|^2)\cdot\nabla M/|\nabla M)^2)=0 \tag{2.48}$$

$$(2\nabla M\cdot\nabla M_t+M_t\Delta M)/a+(\nabla|\nabla M|^2+(\Delta M)\nabla M)\cdot\nabla w_0+(\Delta M)^2$$
$$+(2u_2/c-N/c-\Delta)|\nabla M|^2=0 \tag{2.49}$$

$$\Delta M_t/a+\nabla(\Delta M)\cdot\nabla w_0+(\nabla M\cdot\nabla+2\Delta M)u_2/c-(N/c+\Delta)(\Delta M)=0. \tag{2.50}$$

Finally, we consider system IV. We know from those equations that there are two possibilities. The first one is trivial, that is $w = w_0$, $u = u_2$ and $v = v_2$. The other is similar to that in the case of system I, that is

$$w = H_1 + w_0 \tag{2.51}$$

$$u = F_1 + u_2 \tag{2.52}$$

$$v = F_1 + v_2 \tag{2.53}$$

where H_1 and F_1 satisfy the same equations (2.34) and (2.35) or (2.36) as H and F do. We omit the tedious process of analysis.

3. Analytic solutions

As we know, the homographic transformation is

$$H: M \to \frac{\bar{a}M + \bar{b}}{\bar{c}M + \bar{d}} \, (\bar{a}\bar{d} - \bar{b}\bar{c} \neq 0). \tag{3.1}$$

Under this transformation we introduce two elementary invariants, the Schwarzian derivative and the dimension of velocity, which are

$$S = \{M : x\} = \frac{M_{xxx}}{M_x} - \frac{3}{2}\left(\frac{M_{xx}}{M_x}\right)^2 \tag{3.2}$$

and

$$C = -\frac{M_t}{M_x}. \tag{3.3}$$

The compatibility of (3.2) and (3.3) leads to

$$S_t + C_{xxx} + 2C_x S + CS_x = 0. \tag{3.4}$$

The basic idea of constructing analytic solutions is to turn the Painlevé-Bäcklund equations, such as (2.10)-(2.22), into the polynomials of S, C and their derivatives or sometimes including another important statement $D = -M_{xx}/M_x$ and its derivatives (the so-called Painlevé-Bäcklund transformation), and to make the 'minus square' transformation

$$M_x = V^{-2}. \tag{3.5}$$

Therefore, we can obtain from (3.2) and (3.3) that

$$V_{xx} + SV/2 = 0 \tag{3.6}$$

$$V_t + CV_x = 0. \tag{3.7}$$

This is a linear system and its solutions can be obtained systematically. At this point, we should note that the Painlevé-Bäcklund transformation appears to be correct in many evolution equations according to a series of papers by Weiss *et al* [3, 4, 6, 7].

Hereafter we consider a special case with the dimension 1×1 and use the above method to obtain analytic solutions. Then (2.42)-(2.44) become

$$M_t/b - M_{xx} - M_x w_{0,x} = 0 \tag{3.8}$$

$$(2M_xM_{xt} + M_{xx}M_t)/b - 3M_{xx}M_xw_{0,x} + 2M_x^2v_2/c + NM_x^2/c - M_{xx}^2 - 2M_xM_{xxx} = 0 \tag{3.9}$$

$$M_{xxt}/b - M_{xxx}w_{0,x} + (v_{2,x}M_x + 2v_2M_{xx} + NM_{xx})/c - M_{xxxx} = 0. \tag{3.10}$$

The Painlevé-Bäcklund transformation shows that (3.8) and (3.9) turn into

$$w_{0,x} = -C/b + 2(\log V)_x \tag{3.11}$$

$$v_2 = cC_x/b + cS - N/2 + cD^2/2 \tag{3.12}$$

and (3.10) becomes an identity. We next turn to the original equations and find out that

$$-cV_{0,xx} - v_2 - N = 0 \tag{3.13}$$

$$v_{2,t} - bv_{2,xx} - bw_{0,x}v_{2,x} + bv_2(v_2 + N)/c = 0. \tag{3.14}$$

By substituting (3.11) and (3.12) into (3.13), we know that N should be zero, if it is homographic invariant. Therefore (3.14) becomes

$$\frac{1}{b}(C_{xt} + CC_{xx} + C_x^2) + (S_t - C_{xxx} + CS_x + 2SC_x) - bS_{xx} = 0. \tag{3.15}$$

Particularly, if S and C satisfy the equations

$$C_{xt} + CC_{xx} + C_x^2 = 0 \tag{3.16}$$

$$S_t - C_{xxx} + CS_x + 2SC_x = 0 \tag{3.17}$$

$$S_{xx} = 0 \tag{3.18}$$

then (3.15) holds naturally. The combination (3.16)-(3.18) with (3.4) gives

$$S_{xx} = 0 \tag{31.9}$$

$$C_{xxx} = 0 \tag{3.20}$$

$$C_t + CS_x + 2SC_x = 0 \tag{3.21}$$

$$(C_t + CC_x)_x = 0. \tag{3.22}$$

Clearly, the simplest solution of (3.19)-(3.22) is that S and C are constants. Let $S = -k_0^2/2$ and $C = c_0$. Then (3.6) and (3.7) become

$$V_{xx} - k_0^2V/4 = 0 \tag{3.23}$$

$$V_t + c_0V_x = 0. \tag{3.24}$$

The general solution of this linear system is

$$V = A\,e^{k_0\xi/2} + B\,e^{-k_0\xi/2} \qquad \xi = x - c_0t \tag{3.25}$$

where A and B are arbitrary constants. Then we have from (3.5) that

$$M = \frac{Ce^{k_0\xi/2} + De^{-k_0\xi/2}}{Ae^{k_0\xi/2} + Be^{-k_0\xi/2}} \tag{3.26}$$

where C and D are arbitrary constants provided

$$CB - AD = -1/k_0.$$

So, from (3.11) and (3.12),

$$w_0 = 2\log(A\,\mathrm{e}^{k_0\xi/2} + B\,\mathrm{e}^{-k_0\xi/2}) - \frac{c_0}{b}x + g(t) \tag{3.27}$$

$$v_2 = -\frac{2ck_0^2AB}{(A\,\mathrm{e}^{k_0\xi/2} + B\,\mathrm{e}^{-k_0\xi/2})^2} \tag{3.28}$$

where $g(t)$ is arbitrary function of t.

It follows from the auto-Bäcklund transformation (2.39)-(2.41) that another solution is

$$w = 2\log(C\,\mathrm{e}^{k_0\xi/2} + D\,\mathrm{e}^{-k_0\xi/2}) - \frac{c_0}{b}x + g(t) \tag{3.29}$$

$$v = -\frac{2ck_0^2CD}{(C\,\mathrm{e}^{k_0\xi/2} + D\,\mathrm{e}^{-k_0\xi/2})^2} \tag{3.30}$$

which is just the same as (3.27) and (3.28).

If we choose solutions of system (3.19)-(3.22) other than constants, we can obtain other more complicated analytic solutions. But it is certainly a difficult job.

We can apply this method to system III. The process and the results are similar to those for system II so we omit them. As to system I and system IV, this method is not appropriate since the singularity manifold function M is a constant.

4. Discussion

In this paper, we only discuss the simplified model with constant coefficients μ_u, μ_v and $R(u, v) = 0$. There are still more jobs for us if μ_u and μ_v are not constants or $R(u, v) \neq 0$. On the other hand, how to extend Painlevé-Bäcklund transformations to high-dimensional problems and to construct analytic solutions are more interesting and practical. We will provide further results in a forthcoming publication.

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